

THE GEODESIC RAY TRANSFORM ON TWO-DIMENSIONAL CARTAN-HADAMARD MANIFOLDS

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ABSTRACT. We prove two injectivity theorems for the geodesic ray transform on two-dimensional, complete, simply connected Riemannian manifolds with non-positive Gaussian curvature, also known as Cartan-Hadamard manifolds. The first theorem is concerned with bounded non-positive curvature and the second with decaying non-positive curvature.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In [Hel99] Helgason presents the following result: Suppose that f is a continuous function in \mathbb{R}^2 , $|f(x)| \leq C(1+|x|)^{-\eta}$ for some $\eta > 2$, and $Rf = 0$ where Rf is the Radon transform defined by

$$Rf(x, \omega) := \int_{\mathbb{R}} f(x + t\omega) dt$$

for $x \in \mathbb{R}^2$ and $\omega \in S^1$. Then $f = 0$. Since the operator R is linear this corresponds to the injectivity of the operator. This result was later improved by Jensen [Jen04] requiring that $f = O(|x|^{-\eta})$, $\eta > 1$.

In [Hel94] Helgason presents a similar injectivity result for the hyperbolic 2-space H^2 : Suppose f is a continuous function on H^2 such that $|f(x)| \leq Ce^{-d_g(x,o)}$, where o is a fixed point in H^2 , and

$$\int_{\gamma} f ds = 0$$

for every geodesic γ of H^2 . Then $f = 0$.

The previous results are concerned with constant curvature spaces. There are many related results for Radon type transforms on constant curvature spaces and noncompact homogeneous spaces, see [Hel99],[Hel13]. These types of spaces possess many symmetries. On the other hand, there is also a substantial literature related to geodesic ray transforms on Riemannian manifolds, see e.g. [Muk77], [Sha94], [PSU14]. Here the symmetry assumptions are replaced by curvature or conjugate points conditions, but the spaces are required to be compact with boundary.

In this paper we present injectivity results on two-dimensional, complete, simply connected Riemannian manifolds with non-positive Gaussian curvature. Such manifolds are called Cartan-Hadamard manifolds, and they are

diffeomorphic to \mathbb{R}^2 (hence non-compact) but do not necessarily have symmetries. In order to prove our results we extend energy estimate methods used in [PSU13] to the non-compact case.

Suppose (M, g) is such a manifold and we have a continuous function $f: M \rightarrow \mathbb{R}$. We define the geodesic ray transform $If: SM \rightarrow \mathbb{R}$ of the function f as

$$If(x, v) := \int_{-\infty}^{\infty} f(\gamma_{x,v}(t)) dt,$$

where the unit tangent bundle SM is defined as

$$SM := \{(x, v) \in TM : |v|_g = 1\}$$

and $\gamma_{x,v}$ is the unit speed geodesic with $\gamma_{x,v}(0) = x$ and $\gamma'_{x,v}(0) = v$. Since we are working on non-compact manifolds the geodesic ray transform is not well defined for all continuous functions. We need to impose decay requirements for the functions under consideration. Because of the techniques used we will also impose decay requirements for the first derivatives of the function.

We denote by $C_0(M)$ the set of functions $f \in C(M)$ such that for some $p \in M$ one has $f(x) \rightarrow 0$ as $d(p, x) \rightarrow \infty$. Suppose $p \in M$ and $\eta \in \mathbb{R}$. We define

$$P_\eta(p, M) := \{f \in C(M) : |f(x)| \leq C(1 + d_g(x, p))^{-\eta} \text{ for all } x \in M\},$$

$$P_\eta^1(p, M) := \{f \in C^1(M) : |\nabla f|_g \in P_{\eta+1}(p, M)\} \cap C_0(M).$$

and similarly

$$E_\eta(p, M) := \{f \in C(M) : |f(x)| \leq Ce^{-\eta d_g(x, p)} \text{ for all } x \in M\},$$

$$E_\eta^1(p, M) := \{f \in C^1(M) : |\nabla f|_g \in E_\eta(p, M)\} \cap C_0(M).$$

For all $\eta > 0$ we have inclusions

$$P_\eta^1(p, M) \subset P_\eta(p, M)$$

and

$$E_\eta^1(p, M) \subset E_\eta(p, M),$$

which can be seen by using Lemma 2.1, equation (2.1) and the fundamental theorem of calculus. In addition

$$E_{\eta_1}(p, M) \subset P_{\eta_2}(p, M)$$

for all $\eta_1, \eta_2 > 0$.

We can now state our first injectivity theorem.

Theorem 1. *Suppose (M, g) is a two-dimensional, complete, simply connected Riemannian manifold whose Gaussian curvature satisfies $-K_0 \leq K(x) \leq 0$ for some K_0 . Then the geodesic ray transform is injective on the set $E_\eta^1(M) \cap C^2(M)$ for $\eta > \frac{5}{2}\sqrt{K_0}$.*

The second theorem considers the case of suitably decaying Gaussian curvature. By imposing decay requirements for the Gaussian curvature we are able to relax the decay requirements of the functions we are considering.

Theorem 2. *Suppose (M, g) is a two-dimensional, complete, simply connected Riemannian manifold of non-positive Gaussian curvature K such that $K \in P_{\tilde{\eta}}(p, M)$ for some $\tilde{\eta} > 2$ and $p \in M$. Then the geodesic ray transform is injective on set $P_{\eta}^1(p, M) \cap C^2(M)$ for $\eta > \frac{3}{2}$.*

One question arising is of course the existence of manifolds satisfying the restrictions of the theorems. By the Cartan-Hadamard theorem such manifolds are always diffeomorphic with the plane \mathbb{R}^2 so the question is what kind of Gaussian curvatures we can have on \mathbb{R}^2 endowed with a complete Riemannian metric? The following theorem by Kazdan and Warner [KW74] answers this:

Theorem. *Let $K \in C^\infty(\mathbb{R}^2)$. A necessary and sufficient condition for there to exist a complete Riemannian metric on \mathbb{R}^2 with Gaussian curvature K is that*

$$\lim_{r \rightarrow \infty} \inf_{|x| \geq r} K(x) \leq 0.$$

Especially for every non-positive function $K \in C^\infty(\mathbb{R}^2)$ there exists a metric on \mathbb{R}^2 with Gaussian curvature K .

The case where the metric g differs from the euclidean metric g_0 only in some compact set and the Gaussian curvature is everywhere non-positive is not interesting from the geometric point of view. By a theorem of Green and Gulliver [GG85] if the metric g differs from the euclidean metric g_0 at most on a compact set and there are no conjugate points, then the manifold is isometric to (\mathbb{R}^2, g_0) . Since non-positively curved manifolds can not contain conjugate points this would be the case.

The problem of recovering a function from its integrals over all lines in the plane goes back to Radon [Rad17]. He proved the injectivity of the integral transform nowadays known as the Radon transform and provided a reconstruction formula.

It is also worth mentioning a counterexample for injectivity of the Radon transform provided by Zalzman [Zal82]. He showed that on \mathbb{R}^2 there exists a non-zero continuous function which is $O(|x|^{-2})$ along every line and integrates to zero over any line. See also [AG93], [Arm94].

This work is organized as follows. In the second section we describe the geometrical setting of this work and present some results mostly concerning behaviour of geodesics. The third section is about the geodesic ray transform. In the fourth section we derive estimates for the growth of Jacobi fields in our setting and use those to prove useful decay estimates. The fifth section contains the proofs of our main theorems.

Notational convention. *Throughout this work we denote by $C(a, b, \dots)$ (with a possible subscript) a constant depending on a, b, \dots . The value of the constant may vary from line to line.*

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2. THE SETTING OF THIS WORK AND PRELIMINARIES

Throughout this paper we assume (M, g) to be a two-dimensional, complete, simply connected manifold with non-positive Gaussian curvature K . By the Cartan-Hadamard theorem the exponential map $\exp_x: T_x M \rightarrow M$ is a diffeomorphism for every point $x \in M$. Thereby we have global normal coordinates centered at any point and we could equivalently work with $(\mathbb{R}^2, \tilde{g})$ where \tilde{g} is pullback of the metric g by exponential map, but we choose to present this work in the general setting of (M, g) .

We make the standing assumption of unit-speed parametrization for geodesics. If $x \in M$ and $v \in T_x M$ is such that $|v|_g = 1$ we denote by $\gamma_{x,v}: \mathbb{R} \rightarrow M$ the geodesic with $\gamma_{x,v}(0) = x$ and $\gamma'_{x,v}(0) = v$.

The fact that for every point the exponential map is a diffeomorphism implies that every pair of distinct points can be joined by a unique geodesic. Furthermore, by using the triangle inequality, we have

$$(2.1) \quad d_g(\gamma_{x,v}(t), p) \geq d_g(\gamma_{x,v}(t), x) - d_g(x, p) = |t| - d_g(x, p)$$

for every $p \in M$ and $(x, v) \in SM$.

Because of the everywhere non-positive Gaussian curvature, the function $t \mapsto d_g(\gamma(t), p)$ is convex on \mathbb{R} and the function $t \mapsto d_g(\gamma(t), p)^2$ is strictly convex on \mathbb{R} for every geodesic γ and point $p \in M$ (see e.g. [Pet98]).

We say that the geodesic $\gamma_{x,v}$ is escaping with respect to point p if function $t \mapsto d_g(\gamma_{x,v}(t), p)$ is strictly increasing on the interval $[0, \infty)$. The set of such geodesics is denoted by $\mathcal{E}_p(M)$.

Lemma 2.1. *Let $p \in M$ and $(x, v) \in SM$. At least one of geodesics $\gamma_{x,v}$ and $\gamma_{x,-v}$ is in set $\mathcal{E}_p(M)$.*

Proof. The function $t \mapsto d_g(\gamma(t), p)^2$ is strictly convex on \mathbb{R} so it has a strict global minimum. Therefore the function $t \mapsto d_g(\gamma(t), p)$ also has a strict global minimum, which implies that at least one of functions $t \mapsto d_g(\gamma_{x,v}(t), p)$ and $t \mapsto d_g(\gamma_{x,-v}(t), p)$ is strictly increasing on the interval $[0, \infty)$. \square

If the geodesic $\gamma_{x,v}$ belongs to $\mathcal{E}_p(M)$ equation (2.1) implies the estimate

$$(2.2) \quad d_g(\gamma_{x,v}(t), p) \geq \begin{cases} d_g(x, p), & \text{if } 0 \leq t \leq 2d_g(x, p), \\ t - d_g(x, p), & \text{if } 2d_g(x, p) < t. \end{cases}$$

The manifold M is two-dimensional and oriented and so is also the tangent space $T_x M$ for every $x \in M$. Thus given $v \in T_x M$ we can define $e^{it}v \in T_x M, t \in \mathbb{R}$, to be the unit vector obtained by rotating the vector v by an angle t . We will use the shorthand notation $v_\perp := e^{-i\pi/2}v$.

The unit tangent bundle SM is a 3-dimensional manifold and there is a natural Riemannian metric on it, namely the Sasaki metric [Pat99]. The volume form given by this metric is denoted by $d\Sigma^3$.

On the manifold SM we have the geodesic flow $\varphi_t: SM \rightarrow SM$ defined by

$$\varphi_t(x, v) = (\gamma_{x,v}(t), \gamma'_{x,v}(t)).$$

We denote by X the vector field associated with this flow. We define flows $p_t, h_t: SM \rightarrow SM$ as

$$\begin{aligned} p_t(x, v) &:= (x, e^{it}v), \\ h_t(x, v) &:= (\gamma_{x,v_\perp}(t), Z(t)), \end{aligned}$$

where $Z(t)$ is the parallel transport of the vector v along the geodesic γ_{x,v_\perp} , and denote the associated vector fields by V and X_\perp .

These three vector fields form a global orthonormal frame for $T(SM)$ and we have following structural equations (see [PSU13])

$$\begin{aligned} [X, V] &= X_\perp, \\ [V, X_\perp] &= X, \\ [X, X_\perp] &= -KV, \end{aligned}$$

where K is the Gaussian curvature of the manifold M .

Let $f: U \subset M \rightarrow \mathbb{R}$ be such that $|\nabla f|_g = 1$. Then level sets of the function f are submanifolds of M . The second fundamental form \mathbb{I} on such a level set is defined as

$$\mathbb{I}(v, w) := \text{Hess}(f)(v, w),$$

where $v, w \perp \nabla f$ and $\text{Hess}(f)$ is the covariant Hessian (see [Pet98]).

Suppose that $p \in M$. Denote by $B_p(r)$ the open geodesic ball with radius r , and by $S_p(r)$ its boundary.

Lemma 2.2. *For every $p \in M$ and $r > 0$ the geodesic ball $B_p(r)$ has a strictly convex boundary, i.e. the second fundamental form of $S_p(r)$ is positive definite.*

Proof. Suppose $x \in S_p(r)$ and v is tangent to $S_p(r)$ at x . Denote $f(y) = d_p(y, p)$. We have

$$\text{Hess}(f^2)(x) = 2f(x) \text{Hess} f(x) + 2d_x f \otimes d_x f$$

and thus

$$\text{Hess}(f^2)(x)(v, v) = 2f(x) \text{Hess} f(x)(v, v)$$

since $d_x f(v) = \langle \nabla f(x), v \rangle_g = 0$.

Since the function $t \rightarrow d(\gamma_{x,v}(t), p)^2$ is strictly convex we get

$$\text{Hess}(f^2)(x)(v, v) = \frac{d^2}{dt^2}((f^2 \circ \gamma_{x,v})(t)) \Big|_{t=0} > 0.$$

Therefore $\text{Hess}(f^2)$ is positive definite in tangential directions and so is also $\text{Hess} f$ \square

Equivalently, the boundary of $B_p(r)$ is strictly convex if and only if every geodesics starting from a boundary point in a direction tangent to boundary stays outside $B_p(r)$ for small positive and negative times and has a second order contact at time $t = 0$. From this we see that if $x \in M$ and v is tangent to $S_p(d_g(x, p))$ then function $t \mapsto d_g(\gamma_{x,v}(t), p)^2$ has a global minimum at $t = 0$.

Lemma 2.3. *Suppose $p \in M$ and $(x, v) \in SM$ is such that $\gamma_{x,v} \in \mathcal{E}_p(M)$ and v is not tangent to $S_p(d_g(x, p))$. Then $\gamma_{p_s(x,v)} \in \mathcal{E}_p(M)$ for small s .*

If v is tangent then $\gamma_{p_t(x,v)} \in \mathcal{E}_p(M)$ for either small $t > 0$ or small $t < 0$.

Proof. Suppose first that v is not tangent to $S_p(d_g(x, p))$. Then it must be that

$$\left. \frac{d}{dt} d_g(\gamma_{x,v}(t), p)^2 \right|_{t=0} > 0.$$

The function $s \mapsto \frac{d}{dt} d_g(\gamma_{p_s(x,v)}(t), p)^2$ is continuous and hence

$$(2.3) \quad \left. \frac{d}{dt} d_g(\gamma_{p_s(x,v)}(t), p)^2 \right|_{t=0} > 0$$

for small s . Thus $\gamma_{p_s(x,v)} \in \mathcal{E}_p(M)$.

If v is tangent to $S_p(d_g(x, p))$ then

$$\left. \frac{d}{dt} d_g(\gamma_{x,v}(t), p)^2 \right|_{t=0} = 0$$

and (2.3) holds either for small positive s or for small negative s . □

Lemma 2.4. *Suppose $p \in M$ and $(x, v) \in SM$ is such that $\gamma_{x,v} \in \mathcal{E}_p(M)$. Then $\gamma_{h_s(x,v)} \in \mathcal{E}_p(M)$ for small s .*

Proof. If v is not tangent to $S_p(d_g(x, p))$ then proof is as for the flow p_s . If v is tangent to $S_p(d_g(x, p))$ then $\gamma_{h_s(x,v)}(0)$ is tangent to $S_p(d_g(x, p) + s)$ or $S_p(d_g(x, p) - s)$ and thus $\gamma_{h_s(x,v)} \in \mathcal{E}_p(M)$. □

The next lemma is equation (2.2) for γ_{h_s} and γ_{p_s} .

Lemma 2.5. *For all s such that $\gamma_{h_s(x,v)} \in \mathcal{E}_p(M)$ we have*

$$d_g(\gamma_{h_s(x,v)}(t), p) \geq \begin{cases} d_g(x, p) - s, & 0 \leq t \leq 2d_g(x, p), \\ t - d_g(x, p) - s, & t > 2d_g(x, p). \end{cases}$$

For all s such that $\gamma_{p_s(x,v)} \in \mathcal{E}_p(M)$ we have

$$d_g(\gamma_{p_s(x,v)}(t), p) \geq \begin{cases} d_g(x, p), & 0 \leq t \leq 2d_g(x, p), \\ t - d_g(x, p), & t > 2d_g(x, p). \end{cases}$$

Proof. We have for $\gamma_{h_s(x,v)}$ by triangle inequality

$$d_g(\gamma_{h_s(x,v)}(0), p) \leq d_g(\gamma_{h_s(x,v)}(0), x) + d_g(x, p) = s + d_g(x, p)$$

and furthermore

$$\begin{aligned} d_g(\gamma_{h_s(x,v)}(t), \gamma_{h_s(x,v)}(0)) &\leq d_g(\gamma_{h_s(x,v)}(t), p) + d_g(\gamma_{h_s(x,v)}(0), p) \\ &\leq d_g(\gamma_{h_s(x,v)}(t), p) + s + d_g(x, p). \end{aligned}$$

so

$$t - s - d_g(x, p) \leq d_g(\gamma_{h_s(x,v)}(t), p).$$

By triangle inequality

$$d_g(x, p) \leq d_g(\gamma_{h_s(x,v)}(0), p) + d_g(\gamma_{h_s(x,v)}(0), x) = d_g(\gamma_{h_s(x,v)}(0), p) + s.$$

Because $\gamma_{h_s(x,v)}$ is in $\mathcal{E}_p(M)$ we get for $t \geq 0$

$$d_g(\gamma_{h_s(x,v)}(t), p) \geq d_g(\gamma_{h_s(x,v)}(0), p) \geq d_g(x, p) - s.$$

The result for $\gamma_{h_s(x,v)}$ follows by combining these estimates. For $\gamma_{p_s(x,v)}$ proof is similar, but we have $d_g(\gamma_{p_s(x,v)}(0), x) = 0$. \square

3. THE GEODESIC RAY TRANSFORM

As mentioned in the introduction the geodesic ray transform $If: SM \rightarrow \mathbb{R}$ of a function $f: SM \rightarrow \mathbb{R}$ is defined by

$$If(x, v) := \int_{-\infty}^{\infty} f(\gamma_{x,v}(t)) dt.$$

Lemma 3.1. *The geodesic ray transform is well defined for $f \in P_\eta(p, M)$ for $\eta > 1$.*

Proof. Let $(x, v) \in SM$. Since $If(\gamma_{x,v}(t), \gamma'_{x,v}(t)) = If(x, v)$ for all $t \in \mathbb{R}$, we can assume x to be such that

$$\min_{t \in \mathbb{R}} d_g(\gamma_{x,v}(t), p) = d_g(x, p).$$

Such a point always exists on any geodesic γ since the mapping $t \mapsto d_g(\gamma(t), p)^2$ is strictly convex.

By (2.1) we then have

$$d_g(\gamma_{x,v}(t), p) \geq \begin{cases} d_g(x, p), & \text{if } |t| \leq 2d_g(x, p), \\ |t| - d_g(x, p), & \text{if } 2d_g(x, p) < |t|. \end{cases}$$

Hence for $f \in P_\eta(p, M)$, $\eta > 1$,

$$\begin{aligned} |If(x, v)| &\leq \int_{-\infty}^{\infty} |f(\gamma_{x,v}(t))| dt \leq \int_{-\infty}^{\infty} \frac{C}{(1 + d_g(\gamma_{x,v}(t), p))^\eta} dt \\ &\leq C \left(\int_0^{2d_g(x,p)} \frac{1}{(1 + d_g(x, p))^\eta} dt + \int_{2d_g(x,p)}^{\infty} \frac{1}{(1 + t - d_g(x, p))^\eta} dt \right) \\ &\leq C \left(\frac{2d_g(x, p)}{(1 + d_g(x, p))^\eta} + \frac{1}{(\eta - 1)(1 + d_g(x, p))^{\eta-1}} \right) \\ &\leq \frac{C(\eta)}{(1 + d_g(x, p))^{\eta-1}}. \end{aligned} \quad \square$$

Given a function f on M we define the function $u^f: SM \rightarrow \mathbb{R}$ by

$$u^f(x, v) = \int_0^\infty f(\gamma_{x,v}(t)) dt.$$

We observe that

$$If(x, v) = u^f(x, v) + u^f(x, -v)$$

for all $(x, v) \in SM$ whenever all the functions are well defined.

In the next lemma we assume that f is such that $If \equiv 0$ since those functions are in our interest.

Lemma 3.2. *Suppose $p \in M$ and f is a function on M such that $If \equiv 0$.*

(1) *If $f \in E_\eta(p, M)$ for some $\eta > 0$, then*

$$|u^f(x, v)| \leq C(\eta)(1 + d_g(x, p))e^{-\eta d_g(x, p)}.$$

(2) *If $f \in P_\eta(p, M)$ for some $\eta > 1$, then*

$$|u^f(x, v)| \leq \frac{C(\eta)}{(1 + d(x, p))^{\eta-1}}.$$

Proof. Since $If(x, v) = 0$ we have $|u^f(x, v)| = |u^f(x, -v)|$ for all $(x, v) \in SM$. Thus, by Lemma 2.1, we can assume (x, v) to be such that $\gamma_{x,v} \in \mathcal{E}_p(M)$.

If $f \in P_\eta(p, M)$, $\eta > 1$, using the estimate (2.2) we obtain

$$\begin{aligned} |u^f(x, v)| &\leq C \left(\int_0^{2d_g(x, p)} \frac{1}{(1 + d_g(\gamma_{x,v}(t), p))^\eta} dt \right. \\ &\quad \left. + \int_{2d_g(x, p)}^\infty \frac{1}{(1 + d_g(\gamma_{x,v}(t), p))^\eta} dt \right) \\ &\leq \frac{C(\eta)}{(1 + d_g(x, p))^{\eta-1}}. \end{aligned}$$

Similarly for $f \in E_\eta(p, M)$, $\eta > 0$, we get

$$\begin{aligned} |u^f(x, v)| &\leq C \left(\int_0^{2d_g(x, p)} e^{-\eta d_g(x, p)} dt + \int_{2d_g(x, p)}^\infty e^{-\eta(t - d_g(x, p))} dt \right) \\ &\leq C(\eta)(1 + d_g(x, p))e^{-\eta d_g(x, p)}. \end{aligned} \quad \square$$

Next we prove that $Xu^f = -f$, which can be seen as a reduction to transport equation. This idea is explained in details in [PSU13].

Lemma 3.3. *Suppose $f \in P_\eta^1(p, M)$ for some $\eta > 1$ and $If = 0$. Then $Xu^f(x, v) = -f(x)$ for every $(x, v) \in SM$.*

Proof. We begin by observing that

$$X(If(x, v)) = Xu^f(x, v) + X(u^f(x, -v)) = 0$$

so $Xu^f(x, v) = -X(u^f(x, -v))$. Hence we can assume the geodesic $\gamma_{x,v}$ to be in $\mathcal{E}_p(M)$ by Lemma 2.1.

We have

$$\begin{aligned} Xu^f(x, v) &= \frac{d}{ds} u^f(\varphi_s(x, v)) \Big|_{s=0} = \frac{d}{ds} \int_0^\infty f(\gamma_{\varphi_s(x, v)}(t)) dt \Big|_{s=0} \\ &= \int_0^\infty \frac{d}{ds} f(\gamma_{x, v}(s+t)) \Big|_{s=0} dt \end{aligned}$$

where the last step needs to be justified.

Since we assumed our geodesic to be in $\mathcal{E}_p(M)$, for $t, s \geq 0$ it holds

$$\begin{aligned} \left| \frac{d}{ds} f(\gamma_{x, v}(t+s)) \right| &= |d_{\gamma_{x, v}(t+s)} f(\gamma'_{x, v}(t+s))| \\ &\leq \frac{C}{(1 + d_g(\gamma_{x, v}(t+s), p))^{\eta+1}} \\ &\leq \frac{C}{(1 + d_g(\gamma_{x, v}(t), p))^{\eta+1}}. \end{aligned}$$

Using estimate (2.1) as in the earlier proofs we obtain

$$\begin{aligned} \int_0^\infty \left| \frac{d}{ds} f(\gamma_{x, v}(s+t)) \right| dt &\leq \int_0^\infty \frac{C}{(1 + d_g(\gamma_{x, v}(t), p))^{\eta+1}} dt \\ &\leq \frac{C(\eta)}{(1 + d_g(x, p))^\eta}, \end{aligned}$$

which shows that the last step earlier is justified by the dominated convergence theorem.

Since

$$\frac{d}{ds} f(\gamma_{x, v}(t+s)) \Big|_{s=0} = \frac{d}{dt} f(\gamma_{x, v}(t))$$

and $f(\gamma_{x, v}(t)) \rightarrow 0$ as $t \rightarrow \infty$ we have

$$\int_0^\infty \frac{d}{ds} f(\gamma_{x, v}(s+t)) \Big|_{s=0} dt = -f(x)$$

by the fundamental theorem of calculus. \square

4. REGULARITY AND DECAY OF u^f

In order to prove our main theorems we need to prove C^1 -regularity for u^f given that the function f has suitable regularity and decay properties. For that we derive estimates for functions $X_\perp u^f$ and Vu^f . To prove the estimates for functions $X_\perp u^f$ and Vu^f we will proceed as in the case of $Xu = -f$ (Lemma 3.3). In the proof we calculated

$$\frac{d}{ds} f(\gamma_{\varphi_s(x, v)}(t)) \Big|_{s=0} = d_{\gamma_{\varphi_s(x, v)}(t)} f \left(\frac{d}{ds} \gamma_{\varphi_s(x, v)}(t) \Big|_{s=0} \right).$$

We can interpret $\frac{d}{ds} \gamma_{\varphi_s(x, v)}(t) \Big|_{s=0}$ as a Jacobi field along the geodesic $\gamma_{x, v}$ since it is just the tangent vector field. For $X_\perp u^f$ and Vu^f we proceed in a similar manner, the difference being that the geodesic flow φ_t is replaced with the flows h_t and p_t respectively.

Given geodesic $\gamma_{x,v}$ we denote

$$J_{\gamma_{x,v},h}(s,t) = \frac{d}{dr} \gamma_{h_r(x,v)}(t) \Big|_{r=s}$$

and $J_{\gamma_{x,v},p}$ similarly. Then $J_{\gamma_{x,v},h}(s,t)$ is a Jacobi field along geodesic $\gamma_{h_s(x,v)}$ for fixed s . We will write $J_h(s,t)$ when it is clear from the context what the underlying geodesic is. We will also use shorthand notation $J_h(t) = J_h(0,t)$ and $J_p(t) = J_p(0,t)$.

The Jacobi fields obtained in this manner turn out to be normal with initial data (see [PU04])

$$\begin{aligned} J_h(s,0) &= 1, & D_t J_h(s,0) &= 0, \\ J_p(s,0) &= 0, & D_t J_p(s,0) &= 1. \end{aligned}$$

We need to have estimates for the growth of these two Jacobi fields in particular. The first lemma giving estimates for the growth is based on comparison theorems for Jacobi fields. See for example [Jos08, Theorem 4.5.2].

Lemma 4.1. *Suppose $|K(x)| \leq K_0$ and γ is a geodesic. Then for Jacobi fields J_p and J_h along a geodesic γ it holds that*

$$\begin{aligned} |J_p(t)| &\leq C(K_0) e^{\sqrt{K_0}t}, \\ |J_h(t)| &\leq C(K_0) e^{\sqrt{K_0}t}, \end{aligned}$$

for $t \geq 0$.

This lemma tells us that these Jacobi fields will grow at most exponentially in presence of bounded curvature. If the curvature happens to decay suitably we will see that these Jacobi fields will grow only at a polynomial rate.

If $J(t)$ is a normal Jacobi field along a geodesic γ then we can write $J(t) = u(t)E(t)$ where u is a real valued function and $E(t)$ is a unit normal vector field along γ . From the Jacobi equation it follows that u is a solution to

$$u''(t) + K(\gamma(t))u(t) = 0$$

for $t \geq 0$ with initial values $u(0) = \pm|J(0)|$ and $u'(0) = \pm|D_t J(0)|$.

This leads us to consider an ordinary differential equation

$$(4.1) \quad \begin{cases} u''(t) + K(t)u(t) = 0, & t \geq 0, \\ u(0) = c_1, \\ u'(0) = c_2, \end{cases}$$

for continuous K , where $c_1, c_2 \in \mathbb{R}$. Note that for J_h and J_p the constants c_1 and c_2 are either 0 or ± 1 .

Waltman [Wal64] proved that if u is a solution to (4.1) with K such that

$$\int_0^\infty t|K(t)| ds < \infty$$

then $\lim_{t \rightarrow \infty} u(t)/t$ exists. We reproduce essential parts of the proof in order to obtain a more quantitative estimate for the growth of the solution u .

Lemma 4.2. *Suppose u is a solution to (4.1) with*

$$M_K := \int_0^\infty s|K(s)| ds < \infty.$$

and $c_1 = 1, c_2 = 0$ or other way around. Then

$$|u(t)| \leq C_1 t + C_2$$

for all $t \geq 0$ where $C_1, C_2 \geq 0$.

Proof. We define $A(t) = u'(t)$ and $B(t) = u(t) - tu'(t)$ so $u(t) = A(t)t + B(t)$. Fix $t_0 > 0$. For all $t > t_0$ it holds

$$\begin{aligned} A(t) &= A(t_0) - \int_{t_0}^t K(s)s \left(A(s) + \frac{B(s)}{s} \right) ds, \\ B(t) &= B(t_0) + \int_{t_0}^t K(s)s^2 \left(A(s) + \frac{B(s)}{s} \right) ds. \end{aligned}$$

If we define $|v(t)| = |A(t)| + |B(t)/t|$ we have

$$|v(t)| \leq |v(t_0)| + 2 \int_{t_0}^t s|K(s)||v(s)| ds.$$

By a theorem of Viswanatham [Vis63] it holds $|v(t)| \leq \psi(t)$ on $[t_0, \infty)$ where ψ is a solution to

$$\psi'(t) = 2t|K(t)|\psi(t)$$

with $\psi(t_0) = |v(t_0)|$. Hence

$$\psi(t) = |v(t_0)| e^{2 \int_{t_0}^t s|K(s)| ds} \leq |v(t_0)| e^{2M_K}$$

and furthermore

$$|u(t)| = |tv(t)| \leq te^{2M_K}|v(t_0)|$$

for $t \geq t_0$.

Then we need to estimate $|v(t_0)|$. In order to do so we need estimates for $|u(t_0)|$ and $|u'(t_0)|$. We can apply Lemma 4.1 to get

$$|u(t)| \leq C(K_0)e^{\sqrt{K_0}t_0}$$

on interval $[0, t_0]$ where we have denoted $K_0 = \sup_{t \in [0, t_0]} |K(t)|$. By integrating equation (4.1) we obtain

$$\begin{aligned} |u'(t_0)| &\leq |u'(0)| + \int_0^{t_0} |K(t)||u(t)| ds \\ &\leq |c_2| + \sup_{t \in [0, t_0]} |u(t)| K_0 t_0 \end{aligned}$$

Thus

$$\begin{aligned} |v(t_0)| &\leq |A(t_0)| + |B(t_0)/t_0| \leq |u(t_0)/t_0| + 2|u'(t_0)| \\ &\leq C(K_0)\left(\frac{1}{t_0} + 2K_0 t_0\right)e^{\sqrt{K_0}t_0} + 2. \end{aligned}$$

By combining the estimates for intervals $[0, t_0]$ and $[t_0, \infty)$ and setting $t_0 = 1$ we obtain that

$$|u(t)| \leq te^{2M_K}|v(1)| + C(K_0)e^{\sqrt{K_0}}$$

for $t \geq 0$. \square

Lemma 4.3. *Suppose $|K(x)| \leq K_0$ and that G is a set of geodesics such that*

$$M_G := \sup_{\gamma \in G} \int_0^\infty t|K(t)| dt < \infty.$$

Let $\gamma \in G$. Then for Jacobi fields J_p and J_h along geodesic γ holds

$$\begin{aligned} |J_p(t)| &\leq C(M_G)t, \\ |J_h(t)| &\leq C(M_G)(t+1). \end{aligned}$$

for all $t \geq 0$. Especially the constants do not depend on the geodesic γ .

Proof. Suppose geodesic $\gamma_{x,v}$ is in G . By Lemma 4.2 we obtain

$$\begin{aligned} |J_h(t)| &\leq C_1 t + C_2, \\ |J_p(t)| &\leq C_1 t + C_2. \end{aligned}$$

From the proof of that lemma we see that constants C_1 and C_2 above depend on the lower bound for K and the quantity

$$\int_0^\infty -tK(\gamma_{x,v}(t)) dt.$$

Since this quantity is bounded from above by M_G we can estimate constants C_1 and C_2 by above and get rid of the dependence on the geodesic $\gamma_{x,v}$. So the constants depend only on the Gaussian curvature K and the initial conditions.

Furthermore, since $|J_p(0)| = 0$ we can drop the constant C_2 in the estimate for $J_p(t)$ by making C_1 accordingly larger. \square

Next lemma is a straightforward corollary of the preceding lemma.

Lemma 4.4. *Suppose $K \in P_\eta(p, M)$ for some $\eta > 2$. If $\gamma \in \mathcal{E}_p(M)$ then for Jacobi fields J_p and J_h along geodesic γ one has*

$$\begin{aligned} |J_p(t)| &\leq Ct, \\ |J_h(t)| &\leq C(t+1), \end{aligned}$$

for all $t \geq 0$, where the constants do not depend on the geodesic γ .

Proof. Since $K \in P_\eta(p, M)$, $\eta > 2$, we have

$$\sup_{\gamma \in \mathcal{E}_p(M)} \int_0^\infty -K(\gamma(t))t \, dt < \infty. \quad \square$$

With Lemmas 4.1 and 4.4 we can derive estimates for $X_\perp u^f$ and Vu^f .

Lemma 4.5. *Let $f \in C(M)$ be such that $If = 0$.*

(1) *If $|K(x)| \leq K_0$ and $f \in E_\eta^1(p, M)$ for some $\eta > \sqrt{K_0}$, then*

$$|X_\perp u^f(x, v)| \leq C(\eta, K_0) e^{(2\sqrt{K_0} - \eta)d_g(x, p)}$$

for all $(x, v) \in SM$.

(2) *If $f \in P_\eta^1(p, M)$ for some $\eta > 1$ and $K \in P_{\tilde{\eta}}(p, M)$ for some $\tilde{\eta} > 2$, then*

$$|X_\perp u^f(x, v)| \leq \frac{C(\eta)}{(1 + d_g(x, p))^{\eta-1}}$$

for all $(x, v) \in SM$.

Both estimates hold also if X_\perp is replaced by V .

Proof. Let us first notice that since $If = 0$, it holds $|X_\perp u^f(x, -v)| = |X_\perp u^f(x, v)|$ for all $(x, v) \in SM$. Thus we will assume that v is such that $\gamma_{x,v} \in \mathcal{E}_p(M)$.

First we note that

$$\frac{d}{ds} f(\gamma_{h_s(x,v)}(t)) = d_{\gamma_{h_s(x,v)}(t)} f(J_h(s, t)).$$

By definition

$$\begin{aligned} X_\perp u^f(x, v) &= \frac{d}{ds} \int_0^\infty f(\gamma_{h_s(x,v)}(t)) \, dt \Big|_{s=0} = \int_0^\infty \frac{d}{ds} f(\gamma_{h_s(x,v)}(t)) \, dt \Big|_{s=0} \\ &= \int_0^\infty d_{\gamma_{x,v}(t)} f(J_h(s, t)) \, dt \end{aligned}$$

where the second equality holds by the dominated convergence theorem provided that there exists function $F \in L^1([0, \infty))$ such that

$$(4.2) \quad \left| \frac{d}{ds} f(\gamma_{h_s(x,v)}(t)) \right| \leq F(t)$$

for all $t \geq 0$ and for small non-negative s .

Lemma 2.4 states that for small s it holds that $\gamma_{h_s(x,v)} \in \mathcal{E}_p(M)$. Hence in the first case using Lemmas 2.5 and 4.1 we get

$$\begin{aligned} |d_{\gamma_{h_s(x,v)}(t)} f(J_h(s, t))| &\leq C(K_0) e^{\sqrt{K_0}t} e^{-\eta d_g(\gamma_{h_s(x,v)}(t), p)} \\ &\leq \begin{cases} C(K_0) e^{\eta s} e^{\sqrt{K_0}t} e^{-\eta d_g(x, p)}, & 0 \leq t \leq 2d_g(x, p), \\ C(K_0) e^{\eta s} e^{\sqrt{K_0}t} e^{-\eta(t-d_g(x, p))}, & t > 2d_g(x, p), \end{cases} \end{aligned}$$

and thus

$$\int_0^\infty |d_{\gamma_{h_s(x,v)}(t)} f(J_h(s, t))| \, dt \leq C(\eta, K_0) e^{\eta s} e^{(2\sqrt{K_0} - \eta)d_g(x, p)}.$$

In the second case we obtain

$$|d_{\gamma_{h_s(x,v)}(t)}f(J_h(s,t))| \leq \begin{cases} \frac{C(t+1)}{(1+d_g(x,p)-s)^{\eta+1}}, & 0 \leq t \leq 2d_g(x,p), \\ \frac{C(t+1)}{(1+t-d_g(x,p)-s)^{\eta+1}}, & t > 2d_g(x,p). \end{cases}$$

Therefore

$$\int_0^\infty |d_{\gamma_{h_s(x,v)}(t)}f(J_h(s,t))| dt \leq \frac{C(\eta)}{(1-s+d_g(x,p))^{\eta-1}}.$$

From these estimates we see that such a function F exists in both cases. Setting $s = 0$ gives the estimates for $|X_\perp u^f(x,v)|$.

In case of V instead of X_\perp we proceed in the same manner. First we notice that $|Vu^f(x,-v)| = |Vu^f(x,v)|$ for all $(x,v) \in SM$. Thus we will assume that v is such that $\gamma_{x,v} \in \mathcal{E}_p(M)$. In addition we will assume v to be such that $\gamma_{p_s(x,v)} \in \mathcal{E}_p(M)$ for small non-negative s , this can be done by Lemma 2.3. The rest of the proof is then similar. \square

From this result we see that if f is a C^1 -function with suitable decay properties then u^f is in $C^1(SM)$. Later we will approximate u^f with functions $u^{f_k} \in C^2(SM)$ where functions f_k are compactly supported C^2 -functions on M . The following lemma shows that functions u^{f_k} are indeed in $C^2(SM)$.

Lemma 4.6. *Suppose that $f \in C^2(M)$ is compactly supported. Then $u^f \in C^2(SM)$.*

Proof. Since f is compactly supported we have

$$\begin{aligned} Xu^f(x,v) &= -f(x), \\ X_\perp u^f(x,v) &= \int_0^\infty d_{\gamma_{x,v}(t)}f(J_h(t)) dt, \\ Vu^f(x,v) &= \int_0^\infty d_{\gamma_{x,v}(t)}f(J_p(t)) dt. \end{aligned}$$

From the structural equations and the knowledge that $Xu^f = -f$ we can deduce that $VXu^f, XVu^f, X_\perp Xu^f, XX_\perp u^f$ and X^2u^f exist.

With other means we have to check that $V^2u^f, X_\perp^2u^f$ and $VX_\perp u^f$ (or equivalently $X_\perp Vu^f$) exist.

Let us calculate a formula for $VX_\perp u^f(x,v)$ and from that we see the existence. By definition

$$\begin{aligned} VX_\perp u^f(x,v) &= \frac{d}{ds} X_\perp u^f(p_s(x,v)) \Big|_{s=0} \\ &= \frac{d}{ds} \int_0^\infty d_{\gamma_{p_s(x,v)}(t)}f(J_{\gamma_{p_s(x,v),h}}(t)) dt \Big|_{s=0}. \end{aligned}$$

We write

$$d_{\gamma_{p_s(x,v)}(t)}f(J_{\gamma_{p_s(x,v),h}}(t)) = \langle \nabla f(\gamma_{p_s(x,v)}(t)), J_{\gamma_{p_s(x,v),h}}(t) \rangle.$$

Since

$$\langle D_s \nabla f(\gamma_{p_s(x,v)}(t)), J_{\gamma_{p_s(x,v),h}}(t) \rangle = \text{Hess } f(\gamma_{p_s(x,v)}(t))(J_p(s,t), J_{\gamma_{p_s(x,v),h}}(t))$$

we have

$$\begin{aligned} \frac{d}{ds} d_{\gamma_{ps}(x,v)}(t) f(J_{\gamma_{ps}(x,v),h}(t)) &= \text{Hess } f(\gamma_{ps}(x,v))(J_p(s,t), J_{\gamma_{ps}(x,v),h}(t)) \\ &\quad + \langle \nabla f(\gamma_{ps}(x,v))(t), D_s J_{\gamma_{ps}(x,v),h}(t) \rangle. \end{aligned}$$

Since $\text{Hess } f$ and ∇f are compactly supported we can move derivative $\frac{d}{ds}$ into integral and deduce that $VX_\perp u^f(x,v)$ exists for all $(x,v) \in SM$.

Proofs for $V^2 u^f$ and $X_\perp^2 u^f$ are once again similar. \square

As a last application of Lemmas 4.1 and 4.4 we derive an estimate for the volumes of spheres in our setting.

Lemma 4.7. *Suppose $|K| \leq K_0$ and $p \in M$. Then*

$$\text{Vol } S_p(r) \leq C(K_0) e^{\sqrt{K_0}r}.$$

If $K \in P_\eta(p, M)$ for some $\eta > 2$, then

$$\text{Vol } S_p(r) \leq Ct.$$

Proof. We use polar coordinates centered at point p . Fix a tangent vector $v \in S_p M$ and define mapping $f: [0, \infty) \times (0, 2\pi) \rightarrow M$ by $f(r, \theta) = \exp_p(re^{i\theta}v)$. This gives the usual polar coordinates in which the metric g takes form

$$g(r, \theta) = dr^2 + \left| \frac{df}{d\theta} \right|^2 d\theta^2$$

and the corresponding volume form is

$$dV_g(r, \theta) = \left| \frac{df}{d\theta} \right| dr \wedge d\theta.$$

Since $\exp_p(re^{i\theta}v) = \gamma_{p\theta(p,v)}(r)$ we have

$$\frac{df}{d\theta}(r, \theta) = \frac{d}{dt} \gamma_{p\theta(p,v)}(r) = J_p(r, \theta)$$

and hence the volume form on $S_p(r)$ is given by

$$\iota_{\partial_r} dV_g(r, \theta) = \left| \frac{df}{d\theta} \right| d\theta = J_p(r, \theta) d\theta.$$

By Lemma 4.1

$$\text{Vol } S_p(r) \leq \int_0^{2\pi} C(K_0) e^{\sqrt{K_0}r} d\theta = C(K_0) e^{\sqrt{K_0}r}.$$

In the presence of the additional assumption for the Gaussian curvautre Lemma 4.4 yields

$$\text{Vol } S_p(r) \leq Ct. \quad \square$$

5. PESTOV IDENTITY AND C^2 -APPROXIMATION

In this section we prove our main theorems. The proofs are based on a certain kind of energy estimate for the operator $P = VX$ called the Pestov identity. We will use Pestov identity with boundary terms on submanifolds of (M, g) . Throughout this section we denote $M_{p,r} = B_p(r) \subset M$, a submanifold of M with boundary $S_p(r)$.

The following form of Pestov identity constitutes the main argument for our proofs of the main theorems.

Lemma 5.1 ([IS16]). *For $u \in C^2(SM)$ it holds*

$$\begin{aligned} \|VXu\|_{L^2(SM_{p,r})}^2 &= \|XVu\|_{L^2(SM_{p,r})}^2 + \|Xu\|_{L^2(SM_{p,r})}^2 - \langle KVu, Vu \rangle_{SM_{p,r}} \\ &\quad - \langle \langle v, \nu \rangle Vu, X_\perp u \rangle_{\partial SM_{p,r}} + \langle \langle v_\perp, \nu \rangle Vu, Xu \rangle_{\partial SM_{p,r}} \end{aligned}$$

By using approximating sequences we can relax the regularity assumptions for the Pestov identity. Especially the Pestov identity holds for u^f with suitable f .

Lemma 5.2. *Suppose either one of the following:*

- (1) $|K(x)| \leq K_0$ and $f \in E_\eta^1(p, M) \cap C^2(M)$ for some $\eta > \sqrt{K_0}$.
- (2) $f \in P_\eta^1(p, M) \cap C^2(M)$ for some $\eta > 1$ and $K \in P_{\tilde{\eta}}(p, M)$ for some $\tilde{\eta} > 2$.

If $If = 0$, then the Pestov identity in Lemma 5.1 holds for u^f .

Proof. Lemmas 3.3 and 4.5 ensure that all terms of the Pestov identity are finite.

We define $u_k = u^{\varphi_k f}$ where $\varphi_k: M \rightarrow \mathbb{R}$ is a smooth cutoff function such that

- (1) $0 \leq \varphi_k(x) \leq 1$ for all $x \in M$.
- (2) $\varphi_k(x) = 1$ for $x \in B_p(k)$.
- (3) $\varphi_k(x) = 0$ for $x \notin B_p(2k)$.
- (4) $|\nabla \varphi|_g \leq C/k$ for all $x \in M$ and $v \in T_x M$.

Such a function can be defined by

$$\varphi_k(x) := \varphi\left(\frac{d_g(x, p)}{k}\right)$$

where φ is a suitable smooth cutoff function on \mathbb{R} . Since functions φ_k are smooth and compactly supported, we have $u_k \in C^2(SM)$ by Lemma 4.6.

Let us move on to prove the convergence. First we observe that

$$Xu_k(x, v)|_{SM_{p,r}} = -f(x)$$

for large k . Therefore we have convergence in L^2 -norm for the term Xu_k .

Next we prove convergence for XVu_k under the assumption that $f \in P_\eta^1(p, M) \cap C^2(M)$ for some $\eta > 1$ and $K \in P_{\tilde{\eta}}(p, M)$ for some $\tilde{\eta} > 2$. First we notice that

$$XVu_k = VXu_k + X_\perp u_k = X_\perp u_k$$

for large k . Similarly $XVu^f = X_\perp u^f$ so it is enough to prove that $X_\perp u_k$ converges to $X_\perp u^f$. Furthermore since $SM_{p,r}$ has finite volume it is enough to prove that $X_\perp u_k \rightarrow X_\perp u^f$ in L^∞ -norm.

Let us denote $G = \{\gamma_{x,v} : (x,v) \in SM_{p,r}\}$. The set G fulfills the assumption of Lemma 4.3. Suppose $(x,v) \in SM_r$. We have

$$\begin{aligned} X_\perp u_k(x,v) - X_\perp u^f(x,v) &= \int_0^\infty d_{\gamma_{x,v}(t)}(\varphi_k f)(J_h(t)) dt \\ &\quad - \int_0^\infty d_{\gamma_{x,v}(t)}f(J_h(t)) dt \\ &= \int_0^\infty (\varphi_k(\gamma_{x,v}(t)) - 1)d_{\gamma_{x,v}(t)}f(J_h(t)) dt \\ &\quad + \int_0^\infty f(\gamma_{x,v}(t))d_{\gamma_{x,v}(t)}\varphi_k(J_h(t)) dt. \end{aligned}$$

For $t \geq 0$ holds

$$d_g(\gamma_{x,v}(t), p) \geq t - d_g(x, p) \geq t - r.$$

Also

$$(1 - \varphi_k(\gamma_{x,v}(t))) = 0$$

at least for $0 \leq t \leq k - r$ and $d_{\gamma_{x,v}(t)}\varphi_k$ can be non-zero only in interval $[k - r, 2k + r]$, which can be seen using triangle inequality.

Hence we can estimate, with help of Lemma 4.3, that

$$\begin{aligned} |X_\perp u_k(x,v) - X_\perp u^f(x,v)| &\leq \int_{k-r}^\infty |d_{\gamma_{x,v}(t)}f(J_h(t))| dt \\ &\quad + \int_{k-r}^{2k+r} |f(\gamma_{x,v}(t))d_{\gamma_{x,v}(t)}\varphi_k(J_h(t))| dt \\ &\leq C_1 \int_{k-r}^\infty \frac{t}{(1 + d(\gamma_{x,v}(t), p))^{\eta+1}} dt \\ &\quad + \frac{C_2}{k} \int_{k-r}^{2k+r} \frac{t}{(1 + d(\gamma_{x,v}(t), p))^{\eta+1}} dt \\ &\leq C_1 \int_{k-r}^\infty \frac{t}{(1 + t - r)^{\eta+1}} dt \\ &\quad + \frac{C_2}{k} \int_{k-r}^{2k+r} \frac{t}{(1 + t - r)^{\eta+1}} dt. \end{aligned}$$

The last two integrals do not depend on (x,v) and they also tend to zero as $k \rightarrow \infty$, which proves the L^∞ -convergence. In similar manner we can prove convergence for Vu_k .

Convergence for the boundary terms follows also from the L^∞ -convergence because the boundary $\partial SM_{p,r}$ has a finite volume.

In the other case we proceed similarly but use Lemma 4.1 instead of Lemma 4.3. \square

We are ready to prove our main theorems.

Proof of Theorem 1. Since the geodesic ray transform is linear it is enough to show that $If = 0$ implies $f = 0$.

Let us assume $f \in E_\eta^1(p, M) \cap C^2(M)$, $\eta > \frac{5}{2}\sqrt{K_0}$, is such that $If = 0$. Lemma 5.2 tell us that Pestov identity holds for u^f . We will apply it on submanifold $SM_{p,r}$.

Since $Xu^f = -f$, the term on the left hand side of the Pestov identity is zero. Because we assume Gaussian curvature to be non-positive we have

$$-\langle KVu^f, Vu^f \rangle_{SM_{p,r}} \geq 0.$$

Thus if we can show that the two boundary terms tend to zero as $r \rightarrow \infty$, it must be that

$$\lim_{r \rightarrow 0} \|Xu^f\|_{L^2(SM_{p,r})} = \lim_{r \rightarrow 0} \|f\|_{L^2(SM_{p,r})} = 0$$

which proves the injectivity.

Using Lemma 4.5 together with Lemma 4.7 gives

$$\begin{aligned} |\langle \langle v, \nu \rangle Vu, X_\perp u \rangle_{\partial SM_{p,r}}| &\leq \int_{\partial SM_{p,r}} |Vu^f| |X_\perp u^f| d\Sigma^2 \\ &\leq C(\eta, K_0) \int_{\partial M_{p,r}} \int_{S_x M} e^{2(2\sqrt{K_0}-\eta)d_g(x,p)} dS dV_g \\ &\leq C(\eta, K_0) \int_{\partial M_{p,r}} e^{2(2\sqrt{K_0}-\eta)r} dV_g \\ &\leq C(\eta, K_0) \int_{\partial M_{p,r}} e^{2(2\sqrt{K_0}-\eta)r} dV_g \\ &\leq C(\eta, K_0) e^{2(2\sqrt{K_0}-\eta)r} \text{Vol } S_p(r) \\ &\leq C(\eta, K_0) e^{(5\sqrt{K_0}-2\eta)r}, \end{aligned}$$

which indeed tends to zero as $r \rightarrow \infty$.

Similarly we obtain

$$\left| \int_{\partial SM_{p,r}} \langle v_\perp, \nu \rangle (Vu^f)(Xu^f) d\Sigma^2 \right| \leq C(\eta, K_0) e^{(3\sqrt{K_0}-2\eta)r}.$$

which also tends to zero as $r \rightarrow \infty$. \square

Proof of Theorem 2. The proof is as for the Theorem 1, just using the other estimates provided by Lemmas 4.5 and 4.7. \square

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